The matrix Λ for this case is

$$\Lambda = \begin{bmatrix} J_1(\lambda^2 + 4r_1) & \lambda J_1(1 - r_1) \\ \lambda J_3(r_2 - 1) & J_3(\lambda^2 + r_2) \\ I_1{}^2(\lambda^2 + 4r_3) & \lambda I_1{}^2(1 - r_3) \end{bmatrix}$$

where

$$r_1 = \frac{J_2 - J_3}{J_1}, \qquad r_2 = \frac{J_2 - J_1}{J_3}, \qquad r_3 = \frac{I_2{}^2 - I_3{}^2}{I_1{}^2}$$

For the possibility of asymptotic stability, the rank of Λ must not be less than two for all values of λ. The roots making the determinant of the 2×2 matrix, formed by rows 1 and 2, equal to zero are

$$\lambda^2 = \frac{-(3r_1 + r_1r_2 + 1)}{2} \pm \left\{\left[\frac{3r_1 + r_1r_2 + 1}{2}\right]^2 - 4r_1r_3\right\}^{1/2}$$

The roots of the determinant of the matrix formed by rows 1 and 3 are determined from

$$\lambda[\lambda^2(r_1 - r_3) + 4(r_1 - r_3)] = 0$$

Similarly, for rows 2 and 3,

$$\lambda^2 = \frac{-(3r_3 + r_2r_3 + 1)}{2} \pm \left\{\left[\frac{3r_3 + r_2r_3 + 1}{2}\right]^2 - 4r_2r_3\right\}^{1/2}$$

The only possible condition that is still allowed by Eqs. (10–12), under which the roots of all three of the previous equations are equal, is $r_1 = r_3$. Thus, the rank of the matrix Λ will not be less than two if $r_1 \neq r_3$, or using the definitions of r_1 and r_3, if

$$(I_2{}^1 - I_3{}^1)/I_1{}^1 \neq (I_2{}^2 - I_3{}^2)/I_1{}^2 \qquad (13)$$

The augmented matrix (K',C') is

$$(K',C') = \begin{bmatrix} 4(J_2 - J_3) & 0 & -4(I_2{}^2 - I_3{}^2)b \\ 0 & (J_2 - J_1) & 0 \\ 4(I_2{}^2 - I_3{}^2) & 0 & -[4(I_2{}^2 - I_3{}^2) + k_1]b \end{bmatrix}$$

The rank of the matrix K' is two and the rank of (K',C') can be shown to be not equal to two provided that

$$\frac{k_1}{I_1{}^1} \neq \frac{-4\gamma_1\gamma_2}{[(I_1{}^1/I_1{}^2)\gamma_1 + \gamma_2}$$

which is already not permitted by Eq. (12). Therefore, if the requirements given by Eqs. (10–13) are satisfied the only X^* vector that is a solution of the rotational equations is the trivial solution and the roll-yaw librations are asymptotically stable.

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Does the Center of Flexure Depend on Poisson's Ratio?

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Nomenclature

$Oxyz$	=	orthogonal coordinate system
x,y	=	principal axes of the cross section
F	=	vertical load
$I = \iint x^2dxdy$	=	second moment of area about the y axis
E	=	Young's modulus
ν	=	Poisson's ratio
G	=	modulus of rigidity
y_0	=	coordinate of the flexural center
t_{xz}; t_{yz}	=	shear stress due to torsion in the yz and xz plane, respectively
τ_{xz}; τ_{yz}	=	shear stress due to bending in the yz and xz plane, respectively
$\psi(x,y)$	=	stress function in bending defined by $\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \frac{\nu}{1+\nu}\frac{Fy}{I} - \frac{df(y)}{dy} + C$ and $\frac{\partial\psi}{\partial s} = \left[\frac{Fx^2}{2I} - f(y)\right]\frac{dy}{ds}$ at the boundary
$f(y)$	=	arbitrary function of y only
c; τ	=	constants of integration
ds	=	element of boundary curve
$\varphi(x,y)$	=	stress function in torsion defined by $\partial^2\varphi/\partial x^2 + \partial^2\varphi/\partial y^2 = -2$ and $\varphi = 0$ at the boundary
$\psi_1(x,y)$	=	stress function defined by $\frac{\partial^2\psi_1}{\partial x^2} + \frac{\partial^2\psi_1}{\partial y^2} = \frac{\nu}{1+\nu}\frac{Fy}{I} - \frac{df(y)}{dy} + c$ and $\psi_1 = 0$ at the boundary
$\psi_2(x,y)$	=	stress function defined by $\partial^2\psi_2/\partial x^2 + \partial^2\psi_2/\partial y^2 = 0$ and $d\psi_2 = [Fx^2/2I - f(y)]dy$ at the boundary

Introduction

THE question of the dependence of the center of flexure on Poisson's ratio was explicitly stated first by W. R. Osgood.[1] This question has remained unanswered till now, and there are a great number of recent publications where the center of flexure is determined in one way or another.

We shall develop here a general formula for the position of the flexural center by considering the results obtained previously.[2] The formula obtained shows that the position of the center of flexure does not depend on Poisson's ratio.

The Center of Flexure

The center of flexure is that point at the cross section where the resultant of the shear stresses acts, i.e. where the load is attached. To find the shear center, the shear stresses over the cross section have to be determined by using formulas

$$\tau_{xz} = \partial\psi/\partial y - Fx^2/2I + f(y) \qquad (1)$$

$$\tau_{yz} = -(\partial\psi/\partial x) \qquad (2)$$

in which $f(y)$ is a function of y only and ψ satisfies the equation

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \frac{\nu}{1+\nu}\frac{Fy}{I} - \frac{df(y)}{dy} + c \qquad (3)$$

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at all points of the cross section and the condition

$$\frac{\partial\psi}{\partial s} = \left[\frac{Fx^2}{2I} - f(y)\right]\frac{dy}{ds} \tag{4}$$

at all points of the boundary curve. c is a constant of integration. ds is an element of the boundary curve. F is the load assumed to act vertically on a horizontal cantilever beam. The natural requirement of bending by transversal vertical force is that the resultant moment due to horizontal shear stresses is zero. In this case the resultant of the shear stresses over the cross section is equal to the transversal force which is the actual load applied at the end of the beam. Analytically expressed, this requirement is

$$\iint x\tau_{yz}dxdy = 0 \tag{5}$$

We make use now of the condition (5) and proceed to determine the position of the center of flexure. Since we have already

$$\iint \tau_{yz}dxdy = 0 \tag{6}$$

the resultant shear stresses are in the vertical plane only and satisfy the condition of bending by the vertical transverse load. If for any value of the arbitrary constant c the condition (5) is not fulfilled, we introduce torsional stresses

$$t_{xz} = +G\tau\frac{\partial\varphi}{\partial y}; \quad t_{yz} = -G\tau\frac{\partial\varphi}{\partial x} \tag{7}$$

where the stress function $\varphi(x,y)$ is defined by

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = -2; \quad \varphi_{\text{Boundary}} = 0 \tag{8}$$

and require

$$\iint x(\tau_{yz} + t_{yz})dzdy = 0 \tag{9}$$

The known formula for the position of the center of flexure

$$y_0 = \frac{1}{F}\iint(y\tau_{xz} - x\tau_{yz})dxdy \tag{10}$$

is reduced now to

$$y_0 = \frac{1}{F}\iint y(\tau_{xz} + t_{xz})dxdy \tag{11}$$

From the theory of Saint-Venant's torsion we may write

$$\iint xt_{yz}dxdy = -\iint yt_{xz}dxdy \tag{12}$$

From (9), (11), and (12) we obtain

$$y_0 = \frac{1}{F}\iint(y\tau_{xz} + x\tau_{yz})dxdy \tag{13}$$

With (1) and (2) for τ_{xz} and τ_{yz}, the formula (13) has the form

$$y_0 = \frac{1}{F}\iint\left\{y\frac{\partial\psi}{\partial y} - x\frac{\partial\psi}{\partial x} - \frac{Fyx^2}{2I} + yf(y)\right\}dxdy \tag{14}$$

where ψ is defined by (3) and (4). To simplify the formula (14) we observe the integral

$$T = \iint\left(y\frac{\partial\psi}{\partial y} - x\frac{\partial\psi}{\partial x}\right)dxdy \tag{15}$$

split ψ into $\psi_1 + \psi_2$, and satisfy the conditions (3) and (4) by writing

$$\frac{\partial^2\psi_1}{\partial x^2} + \frac{\partial^2\psi_1}{\partial y^2} = \frac{\nu}{1+\nu}\frac{Fy}{I} - \frac{df(y)}{dy} + c; \quad \psi_{1\text{Boundary}} = 0 \tag{16}$$

$$\frac{\partial^2\psi_2}{\partial x^2} + \frac{\partial^2\psi_2}{\partial y^2} = 0; \quad d\psi_{2\text{Boundary}} = \left(\frac{Fx^2}{2I} - f(y)\right)dy \tag{17}$$

We prove now that

$$T_1 = \iint\left(y\frac{\partial\psi_1}{\partial y} - x\frac{\partial\psi_1}{\partial x}\right)dxdy \tag{18}$$

is equal to zero. We first transform T_1 and obtain

$$T_1 = \int dx\int_y yd\psi_1 - \int dy\int_x xd\psi_1 = \int|y\psi_1|_{y_1}^{y_2}dx - \int|x\psi_1|_{x_1}^{x_2}dy \tag{19}$$

or with $\psi_{1\text{ Boundary}} = 0$, $T_1 = 0$. Equation (14) then changes to

$$y_0 = \frac{1}{F}\iint\left\{y\frac{\partial\psi_2}{\partial y} - x\frac{\partial\psi_2}{\partial x} - \frac{Fyx^2}{2I} + yf(y)\right\}dxdy \tag{20}$$

where ψ_2 is defined by (17).

Conclusion

As we see from Eqs. (20) and (17), the center of flexure is independent from Poisson's ratio.

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Formulas for the Thermodynamic Properties of Dense Nitrogen

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Introduction

FACILITIES such as hypersonic tunnels and gas guns often operate at very high pressures where dense-gas effects must be taken into account in aerodynamic design and performance calculations. The purpose of this Note is to present accurate analytical expressions for the compressibility factor, internal energy, enthalpy, entropy, sound speed, and specific heats of nondissociating, vibrationally excited nitrogen. These properties are obtained using thermodynamic relations with the aid of two carefully selected semiempirical equations of state that apply below and above the critical density, respectively. The constants in these equations of state were chosen to provide the best agreement of the resulting thermodynamic properties with the Arnold Engineering Development Center (AEDC) tables of Grabau and Brahinsky.[1]

Equations of State

Below the critical density, the equation of state employed is a modified Van der Waals equation, which for one mole of

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